UNIFIED TREATMENT OF MULTISYMPLECTIC 3-FORMS IN DIMENSION 6

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Abstract. On a 6-dimensional real vector space V there are three types of multisymplectic 3-forms. We present in this paper a unified treatment of these three types. Forms of each type represent a subset of $\Lambda^3 V^*$. In two cases they are open subsets, in the third one it is a submanifold of codimension 1. We study the geometry of these subsets.

0. Introduction

We shall consider a 6-dimensional real vector space V. Let us recall that a multisymplectic 3-form on V is a 3-form ω such that the associated homomorphism

$$\kappa : V \to \Lambda^2 V^*, \quad \kappa v = \iota_v \omega = \omega(v, \cdot, \cdot)$$

is injective. We denote $\Lambda^3_{ms} V^*$ the subset of $\Lambda^3 V^*$ consisting of all multisymplectic forms. It is easy to see that $\Lambda^3_{ms} V^*$ is an open subset. The natural action of $GL(V)$ on $\Lambda^3 V^*$ preserves $\Lambda^3_{ms} V^*$. It is well known that under this action $\Lambda^3_{ms} V^*$ decomposes into three orbits (see e. g. [D], [H]). Two of them are open orbits, the third one is a submanifold of codimension 1. As representatives of these orbits we can take the following 3-forms. (We choose a basis $e_1, \dots, e_6$ of V, and we denote $\alpha_1, \dots, \alpha_6$ the corresponding dual basis.)

(1) $\omega_+ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,
(2) $\omega_- = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,
(3) $\omega_0 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

The open set containing the form ω_+ (ω_-) we shall denote U_+ (U_-), and the codimension 1 submanifold containing ω_0 we shall denote U_0. There is also another possible characterization of these orbits. Namely, for any 3-form ω we define

$$\Delta^2(\omega) = \{v \in V; (\iota_v \omega) \wedge (\iota_v \omega)\} = 0.$$

In other words, the subset $\Delta^2(\omega) \subset V$ consists of all vectors $v \in V$ such that the 2-form $\iota_v \omega$ is decomposable. A computation shows that

$$\Delta^2(\omega_+) = [e_1, e_2, e_3] \cup [e_4, e_5, e_6],$$
$$\Delta^2(\omega_2) = \{0\},$$
$$\Delta^2(\omega_3) = [e_1, e_2, e_3].$$

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We find easily that

(1) $\omega \in U_+$ if and only if $\Delta^2(\omega)$ consists of the union of two transversal 3-dimensional subspaces.
(2) $\omega \in U_-$ if and only if $\Delta^2(\omega) = \{0\}$.
(3) $\omega \in U_0$ if and only if $\Delta^2(\omega)$ is a 3-dimensional subspace.

We consider now a multisymplectic 3-form ω, and we choose a nonzero 6-form θ on V. It is easy to see that there exists a unique endomorphism $Q : V \to V$ such that

$$(\iota_v \omega) \wedge \omega = \iota_{Qv} \theta. \tag{*}$$

We shall now study the form of the endomorphism Q.

1. The product case

Let us assume that $\omega \in U_+$. Then $\Delta^2(\omega) = V_3' \cup V_3''$, where V_3' and V_3'' are transversal 3-dimensional subspaces. Our main aim in this case is to prove that after the necessary normalization the endomorphism Q is a product structure, i. e. it satisfies $Q^2 = I$, and its associated subspaces are the subspaces V_3' and V_3''.

If $v \in V_3'$, $v \neq 0$ then applying ι_v to (*), we get

$$0 = (\iota_v \omega) \wedge (\iota_v \omega) = \iota_v \iota_{Qv} \theta,$$

which shows that the vectors v and Qv are linearly dependent. This means that there is a function $\lambda_1 : V_3' - \{0\} \to \mathbb{R}$ such that $Qv = \lambda_1(v)v$ for every $v \in V_3' - \{0\}$. It is easy to see that the function λ_1 is constant. Namely, taking two linearly independent vectors $v_1, v_2 \in V_3'$, we get

$$\lambda_1(v_1+v_2)v_1 + \lambda_1(v_1+v_2)v_2 = Q(v_1+v_2) = Q(v_1) + Q(v_2) = \lambda_1(v_1)v_1 + \lambda_1(v_2)v_2,$$

which implies that $\lambda(v_1) = \lambda(v_2)$. Consequently, we have $Qv = \lambda_1 v$ for every $v \in V_3'$. Similarly we find that there is a constant λ_2 such that $Qv = \lambda_2 v$ for every $v \in V_3''$. Now, we are going to prove that $\lambda_1 + \lambda_2 = 0$. We shall need the following lemma.

1.1. Lemma. *If $\omega \in U_+$, $v' \in V_3'$ and $v'' \in V_3''$, then $\iota_{v'} \iota_{v''} \omega = 0$.*

Proof. The lemma is obvious for the form ω_+. But then it holds for every form $\omega \in U_+$.

Let us take two vectors $v' \in V_3'$ and $v'' \in V_3''$, $v' \neq 0$, $v'' \neq 0$. We have

$$(\iota_{v'} \omega) \wedge \omega = \iota_{Qv'} \theta = \lambda_1 \iota_{v'} \theta.$$

Applying $\iota_{v''}$ to the above equation, we get

$$\begin{gathered} (\iota_{v''} \iota_{v'} \omega) \wedge \omega + (\iota_{v'} \omega) \wedge (\iota_{v''} \omega) = \lambda_1 \iota_{v''} \iota_{v'} \theta \\ (\iota_{v'} \omega) \wedge (\iota_{v''} \omega) = \lambda_1 \iota_{v''} \iota_{v'} \theta. \end{gathered}$$

Along the same lines we get

$$\begin{gathered} (\iota_{v''} \omega) \wedge \omega = \iota_{Qv''} \theta = \lambda_2 \iota_{v''} \theta \\ (\iota_{v'} \iota_{v''} \omega) \wedge \omega + (\iota_{v''} \omega) \wedge (\iota_{v'} \omega) = \lambda_2 \iota_{v'} \iota_{v''} \theta \\ (\iota_{v''} \omega) \wedge (\iota_{v'} \omega) = \lambda_2 \iota_{v'} \iota_{v''} \theta. \end{gathered}$$

From the last two results we obtain

$$0 = (\iota_{v'}\omega) \wedge (\iota_{v''}\omega) - (\iota_{v''}\omega) \wedge (\iota_{v'}\omega) = \lambda_1 \iota_{v''}\iota_{v'}\theta - \lambda_2 \iota_{v'}\iota_{v''}\theta = (\lambda_1 + \lambda_2)\iota_{v''}\iota_{v'}\theta,$$

which implies $\lambda_1 + \lambda_2 = 0$. We set now $\lambda = \lambda_1 = -\lambda_2$. Obviously $\lambda \neq 0$. Otherwise we would have $\Delta^2(\omega) = V$, which is a contradiction. Further, we get $Q^2 = \lambda^2 I$. Now we can see that the automorphisms

$$S_+ = \frac{1}{\lambda}Q \text{ and } S_- = -\frac{1}{\lambda}Q \text{ satisfy } S_+^2 = I \text{ and } S_-^2 = I,$$

i. e. they define product structures on V, and $S_- = -S_+$. Setting

$$\theta_+ = \lambda\theta, \quad \theta_- = -\lambda\theta,$$

we get

$$(\iota_v\omega) \wedge \omega = \iota_{S_+v}\theta_+, \quad (\iota_v\omega) \wedge \omega = \iota_{S_-v}\theta_-.$$

In the sequel we shall denote $S = S_+$ and $\theta = \theta_+$. The same results which are valid for S_+ hold also for S_-.

1.2. Lemma. *If $v' \in V_3'$, $v' \neq 0$, then the kernel $K(\iota_{v'}\omega)$ of the 2-form $\iota_{v'}\omega$ equals to $[v', V_3'']$. If $v'' \in V_3''$, $v'' \neq 0$, then the kernel $K(\iota_{v''}\omega)$ of the 2-form $\iota_{v''}\omega$ equals to $[v'', V_3']$.*

Proof. If $v' \in V_3'$, $v' \neq 0$, then the 2-form $\iota_{v'}\omega$ is a nonzero decomposable form. Consequently $\dim K(\iota_{v'}\omega) = 4$. Obviously $v' \in K(\iota_{v'}\omega)$, and by virtue of Lemma 1.1 also any vector from V'' belongs to $K(\iota_{v'}\omega)$. This proves that $K(\iota_{v'}\omega) = [v', V_3'']$. The second assertion follows along the same lines.

1.3. Lemma. *For any $v \in V$ there is $\iota_{Sv}\iota_v\omega = 0$.*

Proof. Let us assume that $S|V_3' = I$ and $S|V_3'' = -I$. Then for arbitrary $v = v' + v''$ with $v' \in V_3'$ and $v'' \in V_3''$ we have

$$\iota_{Sv}\iota_v\omega = \iota_{S(v'+v'')}\iota_{v'+v''}\omega = \iota_{v'-v''}\iota_{v'+v''}\omega = 2\iota_{v'}\iota_{v''}\omega = 0.$$

1.4. Proposition. *There exists a unique (up to the sign) product structure $S \neq I$ on V such that the form ω satisfies the relation*

$$\omega(Sv_1, v_2, v_3) = \omega(v_1, Sv_2, v_3) = \omega(v_1, v_2, Sv_3) \quad \textit{for any } v_1, v_2, v_3 \in V.$$

Proof. We shall prove first that the product structure S defined above satisfies this relation. According to the above lemma we have $\iota_v\iota_{Sv}\omega = 0$ for any $v \in V$. Therefore we have

$$0 = \omega(S(v_1 + v_2), v_1 + v_2, v_3) = \omega(Sv_1, v_2, v_3) + \omega(Sv_2, v_1, v_3),$$

which implies

$$\omega(Sv_1, v_2, v_3) = \omega(v_1, Sv_2, v_3).$$

The second equality now easily follows. Obviously, the opposite product structure $-S$ satisfies the same relation. It remains to prove that there is no other product

structure with the same property. Let $\tilde{S}$ be another product stucture with the above property. Then there is a unique automorphism $A : V \to V$ such that $\tilde{S} = SA$. We have then

$$\begin{aligned}
\omega(v_1, \tilde{S}v_2, v_3) &= \omega(v_1, v_2, \tilde{S}v_3) \\
\omega(v_1, SAv_2, v_3) &= \omega(v_1, v_2, SAv_3) \\
\omega(Sv_1, Av_2, v_3) &= \omega(Sv_1, v_2, Av_3) \\
(\iota_{Sv_1}\omega)(Av_2, v_3) &= (\iota_{Sv_1}\omega)(v_2, Av_3).
\end{aligned}$$

Because S is an automorphism we get the equality

$$(\iota_{v_1}\omega)(Av_2, v_3) = (\iota_{v_1}\omega)(v_2, Av_3).$$

Let us take a vector $v_1' \in V_3'$. Then for any $v_2' \in V_3'$ we have

$$0 = (\iota_{v_1'}\omega)(Av_2', v_1') = (\iota_{v_1'}\omega)(v_2', Av_1').$$

Because v_2' is arbitrary, we can see that Av_1' belongs to the kernel $K(\iota_{v'}\omega)$. This means that there is $\lambda(v_1') \in \mathbb{R}$ and $v'' \in V_3''$ such that $Av_1' = \lambda(v_1')v_1' + v''$. Now we can easily see that there is $\lambda \in \mathbb{R}$ and a homomorphism $\varphi : V_3' \to V_3''$ such that

$$Av_1' = \lambda v_1' + \varphi v_1'$$

for every $v_1' \in V_1'$. Similarly we find $\mu \in \mathbb{R}$ and a homomorphism $\psi : V_3'' \to V_3'$ such that

$$Av_1'' = \mu v_1'' + \psi v_1''$$

for every $v_1'' \in V_3''$. Taking a fixed $v_2' \in V_3'$ and arbitrary $v_1'', v_3'' \in V_3''$, we get

$$\begin{aligned}
(\iota_{v_1''}\omega)(Av_2', v_3'') &= (\iota_{v_1''}\omega)(v_2', Av_3'') \\
(\iota_{v_1''}\omega)(\varphi v_2', v_3'') &= 0, \\
(\iota_{\varphi v_2'}\omega)(v_1'', v_3'') &= 0.
\end{aligned}$$

For any $v_1', v_3' \in V_3'$ we have by virtue of Lemma 1.1

$$(\iota_{\varphi v_2'}\omega)(v_1', v_3'') = 0, \quad (\iota_{\varphi v_2'}\omega)(v_1', v_3') = 0,$$

which together with the preceding result shows that $\iota_{\varphi v_2'}\omega = 0$. The form ω is multisymplectic and consequently $\varphi v_2' = 0$. We have thus shown that $\varphi = 0$. Similarly we find that $\psi = 0$. This proves that $AV_3' \subset V_3'$, $AV_3'' \subset V_3''$ and that $A|V_3' = \lambda I$, $A|V_3'' = \mu I$. Because $\tilde{S}^2 = I$, we find esily that $\lambda = \pm 1$ and $\mu = \pm 1$. Now the proof easily follows.

2. The complex case

In this section we present only the relevant results. Proofs can be found in [PV].

Let ω be a 3-form on V such that $\Delta^2(\omega) = \{0\}$. This means that for any $v \in V$, $v \neq 0$ there is $(\iota_v\omega) \wedge (\iota_v\omega) \neq 0$. This implies that $\operatorname{rank}(\iota_v\omega) \geq 4$. On the other hand obviously $\operatorname{rank}(\iota_v\omega) \leq 4$. Consequently, for any $v \neq 0$ $\operatorname{rank}(\iota_v\omega) = 4$. Thus the kernel $K(\iota_v\omega)$ of the 2-form $\iota_v\omega$ has dimension 2. Moreover $v \in K(\iota_v\omega)$. We have

$$(\iota_v\omega) \wedge \omega = \iota_{Qv}\theta.$$

If $v \neq 0$ then $(\iota_v\omega) \wedge \omega \neq 0$, and this shows that Q is an automorphism. It is also obvious that if $v \neq 0$, then the vectors v and Qv are linearly independent (apply ι_v to the last equality).

2.1. Lemma. *For any $v \in V$ there is $\iota_{Qv}\iota_v\omega = 0$, i. e. $Qv \in K(\iota_v\omega)$.*

This lemma shows that if $v \neq 0$, then $K(\iota_v\omega) = [v, Qv]$. Applying ι_{Qv} to the equality $(\iota_v\omega) \wedge \omega = \iota_{Qv}\theta$ and using the last lemma we obtain easily the following result.

2.2. Lemma. *For any $v \in V$ there is $(\iota_v\omega) \wedge (\iota_{Qv}\omega) = 0$.*

Lemma 2.1 shows that $v \in K(\iota_{Qv}\omega)$. Because v and Qv are linearly independent, we can see that

$$K(\iota_{Qv}\omega) = [v, Qv] = K(\iota_v\omega).$$

It can be proved that that there is $\lambda \in \mathbb{R}$ such that $Q^2 = -\lambda^2 I$. We can now see that the automorphisms

$$J_+ = \frac{1}{\lambda}Q \text{ and } J_- = -\frac{1}{\lambda}Q \text{ satisfy } J_+^2 = -I \text{ and } J_-^2 = -I,$$

i. e. they define complex structures on V, and $J_- = -J_+$. Setting

$$\theta_+ = \lambda\theta, \quad \theta_- = -\lambda\theta$$

we get

$$(\iota_v\omega) \wedge \omega = \iota_{J_+v}\theta_+, \quad (\iota_v\omega) \wedge \omega = \iota_{J_-v}\theta_-.$$

In the sequel we shall denote $J = J_+$ and $\theta = \theta_+$. The same results which are valid for J_+ hold also for J_-.

2.3. Lemma. *There exists a unique (up to the the sign) complex structure J on V such that the form ω satisfies the relation*

$$\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3) = \omega(v_1, v_2, Jv_3) \quad \textit{for any } v_1, v_2, v_3 \in V.$$

3. The tangent case

Let us assume that $\omega \in U_0$. We denote $V_0 = \Delta^2(\omega)$. If $v \in V_0$, $v \neq 0$, then applying ι_v to (*), we get

$$0 = (\iota_v\omega) \wedge (\iota_v\omega) = \iota_v\iota_{Qv}\theta,$$

which shows again that the vectors v and Qv are linearly dependent. Consequently, there exists a function $\lambda : V_0 - \{0\} \to \mathbb{R}$ such that $Qv = \lambda(v)v$ for any $v \in V_0 - \{0\}$. It is easy to see that this function is constant. We shall need the following two lemmas.

3.1. Lemma. *For any $\alpha \in V^*$ we have $(\iota_v\omega) \wedge \omega \wedge \alpha = -\alpha(Qv)\theta$.*

Proof. For a fixed $\alpha \in V^*$ there exists a unique $l_\alpha \in V^*$ such that

$$(\iota_v\omega) \wedge \omega \wedge \alpha = l_\alpha(v)\theta.$$

Hence we get

$$\begin{aligned} (\iota_{Qv}\theta) \wedge \alpha &= l_\alpha(v)\theta \\ \iota_{Qv}(\theta \wedge \alpha) - \alpha(Qv)\theta &= l_\alpha(v)\theta \\ -\alpha(Qv)\theta &= l_\alpha(v)\theta \\ -\alpha(Qv) &= l_\alpha(v), \end{aligned}$$

which finishes the proof.

3.2. Lemma. *Let $\alpha \in V^*$ be such that $\alpha|V_0 = 0$. Then we have $(\iota_v\omega) \wedge \omega \wedge \alpha = 0$.*

Proof. The formula can be verified for the form ω_0 by a direct computation. But then it must be true for any 3-form $\omega \in U_0$.

Using these two lemmas, we get for any 1-form α with $\alpha|V_0 = 0$

$$0 = (\iota_v\omega) \wedge \omega \wedge \alpha = -\alpha(Qv)\theta,$$

which shows that $\alpha(Qv) = 0$. We have thus proved that for any $v \in V$ we have $Qv \in V_0$, i. e. $\operatorname{im} Q \subset V_0$. Further, for any $v \in V$ we have $Q^2v = Q(Qv) = \lambda Qv$. This shows that the endomorphism Q satisfies the equation

$$Q(Q - \lambda I) = 0.$$

Our next aim is to prove that the above constant λ is zero. Let us assume on the contrary that $\lambda \neq 0$. Then there are subspaces $R_0, R_\lambda \subset V$ such that

$$V = R_0 \oplus R_\lambda, \quad Q|R_0 = 0, Q|R_\lambda = \lambda I.$$

Obviously, both these subspaces are nontrivial. $R_0 \neq 0$ because $\ker Q \subset R_0$, and $R_\lambda \neq 0$ because $R_\lambda \supset V_0$. On the other hand for any $v \in R_0$ we have

$$\begin{gathered}(\iota_v\omega) \wedge \omega = \iota_{Qv}\theta = 0\\ (\iota_v\omega) \wedge (\iota_v\omega) = 0.\end{gathered}$$

This shows that $v \in V_0$. Consequently, we get the inclusion $R_0 \subset V_0 \subset R_\lambda$, which is a contradiction. We have thus proved that $\lambda = 0$ and that $Q^2 = 0$. Because for every $v \notin V_0$ we have $Qv \neq 0$ (otherwise we would have $v \in V_0$), it is easy to see that $\operatorname{im} Q = \ker Q = V_0$. The endomorphisms Q satisfying $Q^2 = 0$ are in differential geometry usually called tangent structures, and very often they are denoted by T. But because we would have here already too many T's, we have decided to introduce the notation $F = Q$. We shall call the endomorphism F tangent structure. Let us remark that when speaking about tangent structure, we always assume that $F^2 = 0$ and $\operatorname{im} F = \ker F$.

3.3. Lemma. *For any $v \in V$ we have $\iota_v\iota_{Fv}\omega = 0$.*

Proof. We start with the equality

$$(\iota_v\omega) \wedge \omega = \iota_{Fv}\theta.$$

Applying ι_{Fv} we get

$$\begin{gathered}(\iota_{Fv}\iota_v\omega) \wedge \omega + (\iota_v\omega) \wedge (\iota_{Fv}\omega) = 0\\ -(\iota_v\iota_{Fv}\omega) \wedge \omega + (\iota_v\omega) \wedge (\iota_{Fv}\omega) = 0\\ -\iota_v(\iota_{Fv}\omega \wedge \omega) + 2(\iota_v\omega) \wedge (\iota_{Fv}\omega) = 0.\end{gathered}$$

Applying ι_v we have

$$(\iota_v\omega) \wedge (\iota_v\iota_{Fv}\omega) = 0.$$

If the 1-form $\iota_v\iota_{Fv}\omega$ were not zero, then it would exist a 1-form σ such that $\iota_v\omega = \sigma \wedge \iota_v\iota_{Fv}\omega$, and we would get

$$(\iota_v\omega) \wedge (\iota_v\omega) = \sigma \wedge (\iota_v\iota_{Fv}\omega) \wedge \sigma \wedge (\iota_v\iota_{Fv}\omega) = 0$$

for every $v \in V$, which is a contradiction.

3.4. Lemma. *For any three vectors $v_1, v_2, v_3 \in V$ we have*

$$\omega(Fv_1, v_2, v_3) = \omega(v_1, Fv_2, v_3) = \omega(v_1, v_2, Fv_3).$$

Proof. By virtue of Lemma 3.3 we have

$$0 = \omega(v_1 + v_2, F(v_1 + v_2), v_3) = \omega(v_1, Fv_2, v_3) + \omega(v_2, Fv_1, v_3),$$

which implies

$$\omega(Fv_1, v_2, v_3) = \omega(v_1, Fv_2, v_3).$$

The rest of the proof is easy.

Let us notice that the construction of the tangent struture F depends on the choice of the 6-form θ. Any other nonzero 6-form is a nonzero real multiple $a\theta$ and the relevant construction gives the tangent structure $(1/a)F$. In other words, the 3-form $\omega \in U_0$ determines a tangent structure up to a nonzero real multiple.

We shall now show another possibility how to obtain these tangent structures. It is easy to see that if v, v' are two vectors from the subspace $V_0(\omega_0) = [e_1, e_2, e_3]$, then $\iota_v \iota_{v'} \omega_0 = 0$. Consequently, we have the following lemma.

3.5. Lemma. *Let $\omega \in U_0$. Then for any two vectors $v, v' \in V_0 = \Delta^2(\omega)$ we have $\iota_v \iota_{v'} \omega = 0$.*

3.6. Lemma. *Let $R_3 \subset V$ be a 3-dimensional subspace such that for any two vectors $v, v' \in R_3$ there is $\iota_v \iota_{v'} \omega = 0$. Then $R_3 = \operatorname{im} F$.*

Proof. Let $v, v' \in R_3$. Then we have

$$\begin{aligned}
(\iota_{v'}\omega) \wedge \omega &= \iota_{Fv'}\theta \\
(\iota_v \iota_{v'}\omega) \wedge \omega + (\iota_{v'}\omega) \wedge (\iota_v \omega) &= \iota_v \iota_{Fv'}\theta \\
(\iota_{v'}\omega) \wedge (\iota_v \omega) &= \iota_v \iota_{Fv'}\theta.
\end{aligned}$$

Because the left hand side of this equality is symmetric with respect to v and v', we have

$$\begin{aligned}
\iota_v \iota_{Fv'}\theta &= \iota_{v'} \iota_{Fv}\theta \\
\theta(Fv', v, \cdot, \cdot, \cdot, \cdot) &= \theta(Fv, v', \cdot, \cdot, \cdot, \cdot) \\
\theta(Fv, v', \cdot, \cdot, \cdot, \cdot) &= -\theta(v, Fv', \cdot, \cdot, \cdot, \cdot)
\end{aligned}$$

for any two vectors $v, v' \in R_3$.

Let us assume first that $R_3 \cap \operatorname{im} F$ is 0-dimensional. Then, taking a basis $v_1, v_2, v_3 \in R_3$, we get a basis $v_1, v_2, v_3, Fv_1, Fv_2, Fv_3$ of V, and consequently we have $\theta(v_1, v_2, v_3, Fv_1, Fv_2, Fv_3) \neq 0$. We take the vectors $v_1, v_2, v_3, v_1, Fv_2, Fv_3$. Applying the last formula, we get

$$0 \neq \omega(Fv_1, v_2, v_3, v_1, Fv_2, Fv_3) = -\omega(v_1, Fv_2, v_3, v_1, Fv_2, Fv_3) = 0,$$

which is a contradiction.

Next, let us assume that $R_3 \cap \operatorname{im} F$ is 1-dimensional. Obviously FR_3 is 2-dimensional. Then there are two possibilities. (1) Either $FR_3 \supset R_3 \cap \operatorname{im} F$. Then there are vectors $v_1, v_2 \in R_3$ such that v_1, v_2, Fv_1 is a basis of R_3. Then we can find a vector v_3 such that $v_1, v_2, Fv_1, v_3, Fv_2, Fv_3$ is a basis of V. Taking the vectors $v_1, v_2, v_1, v_3, Fv_2, Fv_3$ and applying the above formula, we get

$$0 \neq \theta(Fv_1, v_2, v_1, v_3, Fv_2, Fv_3) = -\theta(v_1, Fv_2, v_1, v_3, Fv_2, Fv_3) = 0,$$

which is a contradiction. (2) Or $(FR_3) \cap (R_3 \cap \operatorname{im} F) = 0$. Then we can take a basis of R_3 in the form v_1, v_2, Fv_3, and we can complete it to a basis $v_1, v_2, Fv_3, Fv_1, Fv_2, v_3$ of V. This time we take the vectors $v_1, v_2, Fv_3, v_1, Fv_2, v_3$ and we apply the same formula.

$$0 \neq \theta(Fv_1, v_2, Fv_3, v_1, Fv_2, v_3) = -\theta(v_1, Fv_2, Fv_3, v_1, Fv_2, v_3) = 0,$$

which is again a contradiction.

It remains to consider the case when $R_3 \cap \operatorname{im} F$ is 2-dimensional. Then there are again two possibilities. (1) Either $(FR_3) \cap (R_3 \cap \operatorname{im} F) \neq 0$. Then we can take a basis of R_3 in the form v_1, Fv_1, Fv_2, and we can complete it to a basis $v_1, Fv_1, Fv_2, v_2, v_3, Fv_3$. We take the vectors $v_1, v_2, v_1, v_3, Fv_2, Fv_3$ and we apply again the formula.

$$0 \neq \theta(Fv_1, v_2, v_1, v_3, Fv_2, Fv_3) = -\theta(v_1, Fv_2, v_1, v_3, Fv_2, Fv_3) = 0,$$

which is a contradiction. (2) Or $(FR_3) \cap (R_3 \cap \operatorname{im} F) = 0$. Then we take a basis of R_3 in the form v_1, Fv_2, Fv_3, and we complete it to a basis $v_1, Fv_2, Fv_3, Fv_1, v_2, v_3$. Then, taking the vectors $v_1, Fv_2, Fv_3, v_1, v_2, v_3$ we get in the same way as above

$$0 \neq \omega(Fv_1, Fv_2, Fv_3, v_1, v_2, v_3) = -\omega(v_1, F^2v_2, Fv_3, v_1, v_2, v_3) = 0,$$

and we get again a contradiction. In this way we have proved that $R_3 = \operatorname{im} F$.

3.7. Lemma. *Let $\tilde{F} : V \to V$ be a tangent structure (i. e. an endomorphism satisfying $\tilde{F}^2 = 0$ and $\operatorname{im} \tilde{F} = \ker \tilde{F}$) such that*

$$\omega(\tilde{F}v_1, v_2, v_3) = \omega(v_1, \tilde{F}v_2, v_3) = \omega(v_1, v_2, \tilde{F}v_3).$$

Then $\operatorname{im} \tilde{F} = \operatorname{im} F$.

Proof. It suffices to prove that the 3-dimensional subspace $\operatorname{im} \tilde{F}$ has the property described in the preceding lemma. Any two vectors $v, v' \in \operatorname{im} \tilde{F}$ can be expressed in the form $v = \tilde{F}w$, $v' = \tilde{F}w'$. Then we have

$$\iota_v \iota_{v'} \omega = \iota_{\tilde{F}w} \iota_{\tilde{F}w'} \omega = \omega(\tilde{F}w', \tilde{F}w, \cdot) = \omega(\tilde{F}^2 w', w, \cdot) = 0.$$

3.8. Proposition. *Let $\omega \in U_0$. Then there exists (up to a nonzero multiple) a unique tangent structure F such that*

$$\omega(Fv_1, v_2, v_3) = \omega(v_1, Fv_2, v_3) = \omega(v_1, v_2, Fv_3)$$

for all $v_1, v_2, v_3 \in V$.

Proof. Let F and $\tilde{F}$ be two tangent structures with the above property. We introduce on V two 3-forms by setting

$$\sigma_F(v_1, v_2, v_3) = \omega(Fv_1, v_2, v_3), \quad \sigma_{\tilde{F}}(v_1, v_2, v_3) = \omega(\tilde{F}v_1, v_2, v_3).$$

Because by virtue of the preceding lemma $V_0 = \ker F = \ker \tilde{F}$, it is obvious that if $v \in V_0$, then $\iota_v \sigma_F = 0$ and $\iota_v \sigma_{\tilde{F}} = 0$. This implies that there exist two 3-forms s_F and $s_{\tilde{F}}$ on V/V_0 such that

$$\sigma_F = \pi^* s_F, \quad \sigma_{\tilde{F}} = \pi^* s_{\tilde{F}},$$

where $\pi : V \to V/V_0$ is the projection. The tangent structures F and $\tilde{F}$ induce isomorphisms

$$f : V/V_0 \to V_0, \quad \tilde{f} : V/V_0 \to V_0.$$

We denote $A : V/V_0 \to V/V_0$ the automorphism $A = f^{-1}\tilde{f}$. For any three vectors $v_1, v_2, v_3 \in V$ we find

$$s_{\tilde{F}}(\pi v_1, \pi v_2, \pi v_3) = \sigma_{\tilde{F}}(v_1, v_2, v_3) = \omega(\tilde{F}v_1, v_2, v_3) = \omega(\tilde{f}\pi v_1, v_2, v_3).$$

We remind that the last term makes sense because $\tilde{f}\pi v_1 \in V_0$. Further we have

$$\omega(\tilde{f}\pi v_1, v_2, v_3) = \omega(fA\pi v_1, v_2, v_3).$$

Let us choose an element $w_1 \in V$ such that $\pi w_1 = A\pi v_1$. Then we get

$$\begin{aligned} \omega(fA\pi v_1, v_2, v_3) = \omega(f\pi w_1, v_2, v_3) = \omega(Fw_1, v_2, v_3) = \\ = \sigma_F(w_1, v_2, v_3) = s_F(A\pi v_1, \pi v_2, \pi v_3). \end{aligned}$$

Proceeding in this way we obtain the relations

$$\begin{aligned} s_{\tilde{F}}(\pi v_1, \pi v_2, \pi v_3) &= s_F(A\pi v_1, \pi v_2, \pi v_3), \\ s_{\tilde{F}}(\pi v_1, \pi v_2, \pi v_3) &= s_F(\pi v_1, A\pi v_2, \pi v_3), \\ s_{\tilde{F}}(\pi v_1, \pi v_2, \pi v_3) &= s_F(\pi v_1, \pi v_2, A\pi v_3), \end{aligned}$$

and the relation

$$s_F(A\pi v_1, \pi v_2, \pi v_3) = s_F(\pi v_1, A\pi v_2, \pi v_3) = s_F(\pi v_1, \pi v_2, A\pi v_3).$$

Because the 3-form s_F is nontrivial and because the homomorphism $\kappa : V \to \Lambda^2 V^*$ induces an isomorphism $\kappa_0 : V_0 \to \Lambda^2(V/V_0)^*$, we can see that for any 2-form α on V/V_0 and any two vectors $z_1, z_2 \in V/V_0$ we have

$$\alpha(Az_1, z_2) = \alpha(z_1, Az_2).$$

Let now $z \in V/V_0$ be arbitrary, and let us take 1-forms $\beta_1, \beta_2 \in (V/V_0)^*$ such that $\beta_1(z) = \beta_2(z) = 0$. We shall consider the 2-form $\beta_1 \wedge \beta_2$. For any vector $z' \in V/V_0$ we have

$$(\beta_1 \wedge \beta_2)(Az, z') = (\beta_1 \wedge \beta_2)(z, Az') = 0,$$

which shows that there is $\lambda(z) \in \mathbb{R}$ such that $Az = \lambda(z)z$. Moreover, it can be easily seen that the function $\lambda(z)$ is a nonzero constant. We thus get $A = \lambda I$ and this finishes the proof.

Choosing a nonzero 3-form $\eta \in \Lambda^3(V/V_0)^*$, we can define an isomorphism $V/V_0 \to \Lambda^2(V/V_0)^*$ by $w \mapsto \iota_w \eta$. Similarly, the monomorphism $\kappa : V \to \Lambda^2 V^*$, $\kappa v = \iota_v \omega$ induces an isomorphism $\kappa_0 : V_0 \to \Lambda^2(V/V_0)^*$. We take now the following chain of homomorphisms

$$V \xrightarrow{\pi} V/V_0 \to \Lambda^2(V/V_0)^* \xrightarrow{\kappa_0^{-1}} V_0.$$

We denote this composition by C.

3.9. Lemma. *The homomorphism C is a tangent structure satisfying $C^2 = 0$, $\operatorname{im} C = \ker C$ and the relation*

$$\omega(Cv_1, v_2, v_3) = \omega(v_1, Cv_2, v_3) = \omega(v_1, v_2, Cv_3)$$

for every $v_1, v_2, v_3 \in V$.

Proof. Let us take any tangent structure F with the above properties, and let us define a 3-form $\sigma_F(v_1, v_2, v_3) = \omega(Fv_1, v_2, v_3)$ as before. There is a unique 3-form s_F on V/V_0 such that $\sigma_F = \pi^* s_F$, where $\pi : V \to V/V_0$ is the projection. Obviously there is a nonzero $a \in \mathbb{R}$ such that $\eta = as_F$. For any $v, v', v'' \in V$ we have

$$\begin{gathered}\omega(Cv, v', v'') = \eta(\pi v, \pi v', \pi v'') = as_F(\pi v, \pi v', \pi v'') = \\ = a\omega(Fv, v', v'') = \omega(aFv, v', v''),\end{gathered}$$

which shows that $C = aF$. This finishes the proof.

4. Orbit of forms of the product type

This is the orbit U_+, which represents an open submanifold in $\Lambda^3 V^*$. We take a point $\zeta \in U_+$. For the tangent space at this point we have $T_\zeta U_+ = \Lambda^3 V^*$. Obviously, fixing a volume form θ_0 on V, we can choose for each $\zeta \in U_+$ an appropriate volume form $\theta(\zeta)$ (out of the two differring by the sign) such that $\theta(\zeta) = a\theta_0$ with $a > 0$. This means that we choose at the same time at each point $\zeta \in U_+$ a product structure $P(\zeta) \in Aut(V)$. In other words, we can consider over U_+ a trivial vector bundle $\mathcal{V}$ with the fiber V, and on this vector bundle we have a tensor field P of type $(1,1)$ satisfying $P^2 = I$, $\dim\ker(P - I) = 3$, and $\dim\ker(P + I) = 3$. Our aim is to define a product structure on $T_\zeta U_+$. We shall try to define such a product structure by the formula

$$\begin{gathered}(\mathcal{P}(\zeta)\Omega)(v_1, v_2, v_3) = a\Omega(Pv_1, Pv_2, Pv_3) + \\ + b[\Omega(Pv_1, Pv_2, v_3) + \Omega(Pv_1, v_2, Pv_3) + \Omega(v_1, Pv_2, Pv_3)] + \\ + c[\Omega(Pv_1, v_2, v_3) + \Omega(v_1, Pv_2, v_3) + \Omega(v_1, v_2, Pv_3)] + \\ + d\Omega(v_1, v_2, v_3)\end{gathered}$$

for any $\Omega \in T_\zeta U_+$. Here P denotes $P(\zeta)$. It is a matter of computation to prove

4.1. Proposition. *$\mathcal{P}(\zeta)$ satisfies $\mathcal{P}(\zeta)^2 = \mathcal{I}$ if and only if the quadruple (a, b, c, d) is equal to one of the following 16 quadruples*

$$\begin{gathered}(\pm 1, 0, 0, 0), \quad (\pm\tfrac{1}{2}, 0, \mp\tfrac{1}{2}, 0), \quad (0, \pm\tfrac{1}{2}, 0, \mp\tfrac{1}{2}), \quad (0, 0, 0, \pm 1), \\ (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{3}{4}), \quad (\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}), \quad (-\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{3}{4}), \quad (-\tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{3}{4}), \\ (\tfrac{3}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{4}), \quad (\tfrac{3}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}), \quad (-\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}), \quad (-\tfrac{3}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}).\end{gathered}$$

We can define subbundles

$$\mathcal{V}_1 = \ker(P - I), \quad \mathcal{V}_2 = \ker(P + I)$$

satisfying $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. This decomposition enables to introduce in the standard way forms of type (r, s). We denote by the symbol $\mathcal{D}^{r,s}$ the subbundle of the bundle $\Lambda^*\mathcal{V}$ consisting of forms of type (r, s). Now, it is obvious that the tangent bundle of U_+ can be expressed as a direct sum of four subbundles (distributions)

$$TU_+ = \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3},$$

where $\dim \mathcal{D}^{3,0} = \dim \mathcal{D}^{0,3} = 1$, $\dim \mathcal{D}^{2,1} = \dim \mathcal{D}^{1,2} = 9$. Let us denote $\pi_1 : \mathcal{V} \to \mathcal{V}_1$ and $\pi_2 : \mathcal{V} \to \mathcal{V}_2$ the projections. If $\zeta \in U_+$, we can define vectors $\zeta_1, \zeta_2 \in T_\zeta U_+$ by the formulas

$$\zeta_1 = \pi_1^*(\zeta|\mathcal{V}_{1\zeta}), \quad \zeta_2 = \pi_2^*(\zeta|\mathcal{V}_{2\zeta}).$$

Now we can define vector fields ω, ω_1 and ω_2 on U_+ by $\omega_\zeta = \zeta$, $\omega_{1\zeta} = \zeta_1$ and $\omega_{2\zeta} = \zeta_2$. Obviously, $\omega = \omega_1 + \omega_2$.

To each quadruple (a, b, c, d) from Proposition 4.1 there correspond a product structure $\mathcal{P}$ and a subbundle $\mathcal{V}_1 = \ker(\mathcal{P} - \mathcal{I})$. Routine considerations show that the correspondence $(a, b, c, d) \mapsto \mathcal{V}_1$ is the following one.

$$\begin{array}{ll}
(1,0,0,0) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2} & (-1,0,0,0) \mapsto \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3} \\
(\frac{1}{2},0,-\frac{1}{2},0) \mapsto \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} & (-\frac{1}{2},0,\frac{1}{2},0) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \\
(0,\frac{1}{2},0,-\frac{1}{2}) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{0,3} & (0,-\frac{1}{2},0,\frac{1}{2}) \mapsto \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \\
(0,0,0,1) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} & (0,0,0,-1) \mapsto 0 \\
(\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4}) \mapsto \mathcal{D}^{3,0} & (\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{3}{4}) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \\
(-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{3}{4}) \mapsto \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} & (-\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{3}{4}) \mapsto \mathcal{D}^{0,3} \\
(\frac{3}{4},\frac{1}{4},-\frac{1}{4},\frac{1}{4}) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} & (\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}) \mapsto \mathcal{D}^{1,2} \\
(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3} & (-\frac{3}{4},-\frac{1}{4},\frac{1}{4},-\frac{1}{4}) \mapsto \mathcal{D}^{2,1}
\end{array}$$

In the sequel we are going to investigate the integrability of all these distributions. Our first result is easy because the distributions $\mathcal{D}^{3,0}$ and $\mathcal{D}^{0,3}$ are 1-dimensional.

4.2. Proposition. *The distribution $\mathcal{D}^{3,0}$ ($\mathcal{D}^{3,0}$) is generated by the vector field ω_1 (ω_2). The distributions $\mathcal{D}^{3,0}$ and $\mathcal{D}^{0,3}$ are integrable.*

Now we shall introduce on U_+ a flat connection ∇, which is the restriction of the canonical connection on the vector space $\Lambda^3 V^*$. Notice that for any vector field Ω on U_+ we have $\nabla_\Omega \omega = \Omega$. We shall need the following three lemmas.

4.3. Lemma. *Let $\tilde{\Omega}$ be a vector field on U_+ belonging to $\mathcal{D}^{3,0}$ ($\mathcal{D}^{2,1}$, $\mathcal{D}^{1,2}$, $\mathcal{D}^{0,3}$). Further, let Ω be arbitrary vector field on U_+. Then*

$$\nabla_\Omega \tilde{\Omega} \in \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \quad (\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}, \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}, \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}).$$

Proof. Let Θ be a section of the trivial vector bundle $\mathcal{V}^*$ over U_+. Then for any vector field Ω on U_+ we have

$$\nabla_\Omega \Theta \in \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}.$$

Now the assertion of the lemma easily follows.

4.4. Lemma. *If Ω belongs to the distribution $\mathcal{D}^{3,0}$ ($\mathcal{D}^{0,3}$), then we have*

$$\nabla_\Omega \omega_1 = \Omega, \quad \nabla_\Omega \omega_2 = 0 \qquad (\nabla_\Omega \omega_1 = 0, \quad \nabla_\Omega \omega_2 = \Omega).$$

If Ω belongs to the distribution $\mathcal{D}^{2,1}$ ($\mathcal{D}^{1,2}$), then we have again

$$\nabla_\Omega \omega_1 = \Omega, \quad \nabla_\Omega \omega_2 = 0 \qquad (\nabla_\Omega \omega_1 = 0, \quad \nabla_\Omega \omega_2 = \Omega).$$

Proof. We start with the equality $\omega_1 + \omega_2 = \omega$. If Ω belongs to $\mathcal{D}^{3,0}$, then applying ∇_Ω to this equality we get

$$\nabla_\Omega \omega_1 + \nabla_\Omega \omega_2 = \Omega$$
$$(\nabla_\Omega \omega_1)^{3,0} + (\nabla_\Omega \omega_1)^{2,1} + (\nabla_\Omega \omega_2)^{1,2} + (\nabla_\Omega \omega_2)^{0,3} = \Omega,$$

where the superscripts denote the corresponding component. Because Ω belongs to $\mathcal{D}^{3,0}$ we obtain the first assertion. The remaining assertions follow along the same lines.

4.5. Lemma. *A vector field Ω belongs to the distribution $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}$ ($\mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$) if and only if*

$$\nabla_\Omega \omega_2 = 0 \quad (\nabla_\Omega \omega_1 = 0).$$

Proof. If Ω belongs to $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}$ we know that the above condition is satisfied. Conversely, let us assume that the condition is satisfied. We have

$$\Omega = \Omega^{3,0} + \Omega^{2,1} + \Omega^{1,2} + \Omega^{0,3},$$

and we get

$$0 = \nabla_\Omega \omega_2 = \nabla_{\Omega^{3,0}} \omega_2 + \nabla_{\Omega^{2,1}} \omega_2 + \nabla_{\Omega^{1,2}} \omega_2 + \nabla_{\Omega^{0,3}} \omega_2 =$$
$$= \nabla_{\Omega^{1,2}} \omega_2 + \nabla_{\Omega^{0,3}} \omega_2 = \Omega^{1,2} + \Omega^{0,3},$$

which finishes the proof.

4.6. Proposition. *The distributions $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}$ and $\mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$ are integrable.*

Proof. Let two vector fields $\Omega, \tilde{\Omega}$ belong to the distribution $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}$. Then we have $\nabla_\Omega \omega_2 = \nabla_{\tilde{\Omega}} \omega_2 = 0$, and we obtain

$$\nabla_{[\Omega,\tilde{\Omega}]} \omega_2 = \nabla_\Omega \nabla_{\tilde{\Omega}} \omega_2 - \nabla_{\tilde{\Omega}} \nabla_\Omega \omega_2 = 0$$

because the connection ∇ is flat. Along the same lines we can prove the integrability of the distribution $\mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$.

The following lemma is obvious.

4.7. Lemma. *A vector field Ω belongs to the distribution $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$ if and only if $\Omega \wedge \omega = 0$.*

4.8. Proposition. *The distribution $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$ is not integrable.*

Proof. Let Ω and $\tilde{\Omega}$ lie in $\mathcal{D}^{2,1}$ and $\mathcal{D}^{1,2}$, respectively. Then we have $\Omega \wedge \omega = 0$ and $\tilde{\Omega} \wedge \omega = 0$. Hence we obtain

$$(\nabla_\Omega \tilde{\Omega}) \wedge \omega + \tilde{\Omega} \wedge \Omega = 0, \quad (\nabla_{\tilde{\Omega}} \Omega) \wedge \omega + \Omega \wedge \tilde{\Omega} = 0.$$

Substracting these two equalities, we have

$$[\Omega, \tilde{\Omega}] \wedge \omega = 2\Omega \wedge \tilde{\Omega}.$$

Now it suffices to choose Ω and $\tilde{\Omega}$ in such a way that $\Omega_\zeta \wedge \tilde{\Omega}_\zeta \neq 0$ at some point $\zeta \in U_+$. Then it is obvious that the bracket $[\Omega, \tilde{\Omega}]$ does not lie in $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$.

4.9. Proposition. *The distributions $\mathcal{D}^{2,1}$ and $\mathcal{D}^{1,2}$ are integrable.*

Proof. Let Ω and $\tilde{\Omega}$ be two vector fields lying in $\mathcal{D}^{2,1}$. Proceeding in the same way as in the proof of preceding lemma we find again

$$[\Omega, \tilde{\Omega}] \wedge \omega = 2\Omega \wedge \tilde{\Omega}.$$

But this time $\Omega \wedge \tilde{\Omega} = 0$, which shows that $[\Omega, \tilde{\Omega}]$ lies in $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$. Moreover, we have

$$\nabla_{[\Omega, \tilde{\Omega}]} \omega_2 = \nabla_\Omega \nabla_{\tilde{\Omega}} \omega_2 - \nabla_{\tilde{\Omega}} \nabla_\Omega \omega_2 = 0,$$

which shows that $[\Omega, \tilde{\Omega}]$ lies in $\mathcal{D}^{2,1}$.

4.10. Proposition. *There is $[\omega_1, \omega_2] = 0$ and the distribution $\mathcal{D}^{3,0} \oplus \mathcal{D}^{0,3}$ is integrable.*

Proof. We have

$$\nabla_{[\omega_1, \omega_2]} \omega_1 = \nabla_{\omega_1} \nabla_{\omega_2} \omega_1 - \nabla_{\omega_2} \nabla_{\omega_1} \omega_1 = 0 - \nabla_{\omega_2} \omega_1 = 0,$$

which shows that $[\omega_1, \omega_2]$ lies in $\mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$. Along the same lines we can show that $[\omega_1, \omega_2]$ lies in $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}$. This implies that $[\omega_1, \omega_2] = 0$ and that the distribution $\mathcal{D}^{3,0} \oplus \mathcal{D}^{0,3}$ is integrable.

4.11. Proposition. *For any vector field Ω lying in $\mathcal{D}^{1,2}$ ($\mathcal{D}^{2,1}$) the vector field $[\omega_1, \Omega]$ ($[\omega_2, \Omega]$) lies again in $\mathcal{D}^{1,2}$ ($\mathcal{D}^{2,1}$). Consequently the distributions $\mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2}$ and $\mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3}$ are integrable.*

Proof. Let us assume that Ω lies in $\mathcal{D}^{1,2}$. Then we have

$$\nabla_{[\omega_1, \Omega]} \omega_1 = \nabla_{\omega_1} \nabla_\Omega \omega_1 - \nabla_\Omega \nabla_{\omega_1} \omega_1 = 0 - \nabla_\Omega \omega_1 = 0,$$

which proves that $[\omega_1, \Omega]$ lies in $\mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$. Because Ω lies in $\mathcal{D}^{1,2}$, there is $\Omega \wedge \omega = 0$. Applying ∇_{ω_1} to this equality we find that

$$0 = (\nabla_{\omega_1} \Omega) \wedge \omega + \Omega \wedge \nabla_{\omega_1} \omega = (\nabla_{\omega_1} \Omega) \wedge \omega + \Omega \wedge \omega_1.$$

Obviously $\Omega \wedge \omega_1 = 0$, and this shows that $\nabla_{\omega_1} \Omega$ lies in $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$. But we can immediately see that

$$[\omega_1, \Omega] = \nabla_{\omega_1} \Omega - \nabla_\Omega \omega_1 = \nabla_{\omega_1} \Omega.$$

Consequently $[\omega_1, \Omega]$ lies not only in $\mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$, but also in $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$. This implies that $[\omega_1, \Omega]$ lies in $\mathcal{D}^{1,2}$.

4.12. Proposition. *The distributions $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3}$ and $\mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$ are integrable. The distributions $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$ and $\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$ are not integrable.*

Proof. The first assertion is easy to prove. Therefore, let us consider the distribution $\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$. We shall take the same vector fields Ω lying in $\mathcal{D}^{2,1}$ and $\tilde{\Omega}$ lying in $\mathcal{D}^{1,2}$ as in the proof of Proposition 4.8. Then we have

$$\begin{gathered}
[\Omega, \tilde{\Omega}] \wedge \omega_1 = (\nabla_\Omega \tilde{\Omega}) \wedge \omega_1 - (\nabla_{\tilde{\Omega}} \Omega) \wedge \omega_1 = \\
= \nabla_\Omega (\tilde{\Omega} \wedge \omega_1) - \tilde{\Omega} \wedge (\nabla_\Omega \omega_1) - \nabla_{\tilde{\Omega}} (\Omega \wedge \omega_1) + \Omega \wedge (\nabla_{\tilde{\Omega}} \omega_1) = \\
= -\tilde{\Omega} \wedge \Omega = \Omega \wedge \tilde{\Omega}.
\end{gathered}$$

At the same point $\zeta \in U_+$ as in the proof of Proposition 4.8 we have $\Omega_\zeta \wedge \tilde{\Omega}_\zeta \neq 0$, which shows that $[\Omega, \tilde{\Omega}]_\zeta^{0,3} \neq 0$. This proves that the distribution under consideration is not integrable.

We can summarize our results.

4.13. Proposition. *The distributions*

$$\begin{gathered}
\mathcal{D}^{3,0}, \quad \mathcal{D}^{2,1}, \quad \mathcal{D}^{1,2}, \quad \mathcal{D}^{0,3} \\
\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}, \quad \mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2}, \quad \mathcal{D}^{3,0} \oplus \mathcal{D}^{0,3}, \quad \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3}, \quad \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} \\
\mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3}, \quad \mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}
\end{gathered}$$

are integrable. The distributions

$$\mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}, \quad \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}, \quad \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}$$

are not integrable.

4.14. Remark. Requiring $\dim \ker(\mathcal{P} - \mathcal{I}) = \dim \ker(\mathcal{P} + \mathcal{I}) = 10$ we have only four possibilities how to define a product structure $\mathcal{P}$. It is easy to see that these product structures correspond to the quadruples

$$(1,0,0,0), \quad (-1,0,0,0), \quad (\frac{1}{2}, 0, -\frac{1}{2}, 0), \quad (-\frac{1}{2}, 0, \frac{1}{2}, 0).$$

Because all the distributions associated with these projectors are integrable, in all these cases the Nijenhuis tensor $[\mathcal{P}, \mathcal{P}] = 0$.

5. Orbit of forms of the complex type

Here we shall study the orbit U_-, which also represents an open submanifold in $\Lambda^3 V^*$. Taking a point $\zeta \in U_-$, we have $T_\zeta U_- = \Lambda^3 V^*$. Fixing again a volume form θ_0 on V, we can choose for each $\zeta \in U_-$ an appropriate volume form $\theta(\zeta)$ (out of the two differring by the sign) such that $\theta(\zeta) = a\theta_0$ with $a > 0$. This enables us to choose at each point $\zeta \in U_-$ a complex structure $J(\zeta) \in Aut(V)$. In other words, this time we have on the trivial vector bundle $\mathcal{V}$ a tensor field J of type $(1,1)$

satisfying $J^2 = -I$. We shall again try to define a complex structure on $T_\zeta U_-$ by the formula

$$(\mathcal{J}(\zeta)\Omega)(v_1, v_2, v_3) = a\Omega(Jv_1, Jv_2, Jv_3)+$$
$$+b[\Omega(Jv_1, Jv_2, v_3) + \Omega(Jv_1, v_2, Jv_3) + \Omega(v_1, Jv_2, Jv_3)]+$$
$$+c[\Omega(Jv_1, v_2, v_3) + \Omega(v_1, Jv_2, v_3) + \Omega(v_1, v_2, Jv_3)]+$$
$$+d\Omega(v_1, v_2, v_3)$$

for any $\Omega \in T_\zeta U_-$.

5.1. Proposition. *$\mathcal{J}(\zeta)$ satisfies $\mathcal{J}(\zeta)^2 = -\mathcal{I}$ if and only if the quadruple (a, b, c, d) is equal to one of the following 4 quadruples*

$$(\pm 1, 0, 0, 0), \quad (\pm\frac{1}{2}, 0, \pm\frac{1}{2}, 0).$$

The proof is a simple computation and will be omitted. We shall denote

$$(\mathcal{J}_1(\zeta)\Omega)(v_1, v_2, v_3) = \Omega(J(\zeta)v_1, J(\zeta)v_2, J(\zeta)v_3)$$
$$(\mathcal{J}_2(\zeta)\Omega)(v_1, v_2, v_3) = \frac{1}{2}\Omega(J(\zeta)v_1, J(\zeta)v_2, J(\zeta)v_3)+$$
$$+\frac{1}{2}[\Omega(J(\zeta)v_1, v_2, v_3) + \Omega(v_1, J(\zeta)v_2, v_3) + \Omega(v_1, v_2, J(\zeta)v_3)].$$

The mapping $\zeta \in U_- \mapsto J_1(\zeta)$ (resp. $\zeta \in U_- \mapsto J_2(\zeta)$) defines an almost complex structure $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) on the orbit U_-.

5.2. Proposition. *The almost complex structure $\mathcal{J}_2$ is integrable.*

Proof. We denote again by ∇ the canonical connection on $\Lambda^3 V^*$. Let Ω and $\tilde{\Omega}$ be two vector fields on U_-. Applying $\nabla_{\tilde{\Omega}}$ to the identity $J^2 = -I$, we get

$$(\nabla_{\tilde{\Omega}} J)J + J(\nabla_{\tilde{\Omega}} J) = 0.$$

Further, we shall use the identity

$$\omega(Jv_1.v_2, v_3) = \omega(v_1, Jv_2, v_3),$$

and apply to it the covariant derivative $\nabla_{\tilde{\Omega}}$. We obtain

$$\tilde{\Omega}(Jv_1, v_2, v_3) + \omega((\nabla_{\tilde{\Omega}} J)v_1, v_2, v_3) = \tilde{\Omega}(v_1, Jv_2, v_3) + \omega(v_1, (\nabla_{\tilde{\Omega}} J)v_2, v_3).$$

Substituing now Jv_2 instead of v_2 and $(\nabla_\Omega J)v_3$ instead of v_3, we get the relation

$$\tilde{\Omega}(Jv_1, Jv_2, (\nabla_\Omega J)v_3) =$$
$$-\tilde{\Omega}(v_1, v_2, (\nabla_\Omega J)v_3) - \omega((\nabla_{\tilde{\Omega}} J)v_1, Jv_2, (\nabla_\Omega J)v_3) - \omega(Jv_1, (\nabla_{\tilde{\Omega}} J)v_2, (\nabla_\Omega J)v_3).$$

Similarly we obtain the relations

$$\tilde{\Omega}(Jv_1, (\nabla_\Omega J)v_2, Jv_3) =$$
$$-\tilde{\Omega}(v_1, (\nabla_\Omega J)v_2, v_3) - \omega(Jv_1, (\nabla_\Omega J)v_2, (\nabla_{\tilde{\Omega}} J)v_3) - \omega((\nabla_{\tilde{\Omega}} J)v_1, (\nabla_\Omega J)v_2, Jv_3),$$
$$\tilde{\Omega}((\nabla_\Omega J)v_1, Jv_2, Jv_3) =$$
$$-\tilde{\Omega}((\nabla_\Omega J)v_1, v_2, v_3) - \omega((\nabla_\Omega J)v_1, (\nabla_{\tilde{\Omega}} J)v_2, Jv_3) - \omega((\nabla_\Omega J)v_1, Jv_2, (\nabla_{\tilde{\Omega}} J)v_3).$$

Let us compute now

$$\begin{gathered}
2(\nabla_\Omega(\mathcal{J}\tilde{\Omega}))(v_1, v_2, v_3) = 2\nabla_\Omega((\mathcal{J}\tilde{\Omega})(v_1, v_2, v_3)) = \nabla_\Omega(\tilde{\Omega}(Jv_1, Jv_2, Jv_3) + \\
+[\tilde{\Omega}(Jv_1, v_2, v_3) + \tilde{\Omega}(v_1, Jv_2, v_3) + \tilde{\Omega}(v_1, v_2, Jv_3)]) = (\nabla_\Omega\tilde{\Omega})(Jv_1, Jv_2, Jv_3) + \\
+(\nabla_\Omega\tilde{\Omega})(Jv_1, v_2, v_3) + (\nabla_\Omega\tilde{\Omega})(v_1, Jv_2, v_3) + (\nabla_\Omega\tilde{\Omega})(v_1, v_2, Jv_3) + \\
+\tilde{\Omega}((\nabla_\Omega J)v_1, Jv_2, Jv_3) + \tilde{\Omega}(Jv_1, (\nabla_\Omega J)v_2, Jv_3) + \tilde{\Omega}(Jv_1, Jv_2, (\nabla_\Omega J)v_3) + \\
+\tilde{\Omega}((\nabla_\Omega J)v_1, v_2, v_3) + \tilde{\Omega}(v_1, (\nabla_\Omega J)v_2, v_3) + \tilde{\Omega}(v_1, v_2, (\nabla_\Omega J)v_3) = \\
= 2(\mathcal{J}\nabla_\Omega\tilde{\Omega})(v_1, v_2, v_3) - \\
-\omega((\nabla_\Omega J)v_1, (\nabla_{\tilde{\Omega}} J)v_2, Jv_3) - \omega((\nabla_\Omega J)v_1, Jv_2, (\nabla_{\tilde{\Omega}} J)v_3) \\
-\omega(Jv_1, (\nabla_\Omega J)v_2, (\nabla_{\tilde{\Omega}} J)v_3) - \omega((\nabla_{\tilde{\Omega}} J)v_1, (\nabla_\Omega J)v_2, Jv_3) \\
-\omega((\nabla_{\tilde{\Omega}} J)v_1, Jv_2, (\nabla_\Omega J)v_3) - \omega(Jv_1, (\nabla_{\tilde{\Omega}} J)v_2, (\nabla_\Omega J)v_3).
\end{gathered}$$

Here we have used the previous relations. Let us notice that the expression consisting of the last six terms is symmetric with respect to Ω and $\tilde{\Omega}$. Consequently we obtain

$$\nabla_\Omega(\mathcal{J}\tilde{\Omega}) - \nabla_{\tilde{\Omega}}(\mathcal{J}\Omega) = \mathcal{J}(\nabla_\Omega\tilde{\Omega} - \nabla_{\tilde{\Omega}}\Omega) = \mathcal{J}[\Omega, \tilde{\Omega}].$$

Writing $\mathcal{J}\Omega$ instead of Ω, we get

$$\nabla_{\mathcal{J}\Omega}(\mathcal{J}\tilde{\Omega}) = -\nabla_{\tilde{\Omega}}\Omega + \mathcal{J}[\mathcal{J}\Omega, \tilde{\Omega}].$$

Interchanging Ω and $\tilde{\Omega}$ we get the relation

$$\nabla_{\mathcal{J}\tilde{\Omega}}(\mathcal{J}\Omega) = -\nabla_\Omega\tilde{\Omega} + \mathcal{J}[\mathcal{J}\tilde{\Omega}, \Omega].$$

Substracting these last two relations we obtain

$$[\mathcal{J}\Omega, \mathcal{J}\tilde{\Omega}] = [\Omega, \tilde{\Omega}] + \mathcal{J}[\mathcal{J}\Omega, \tilde{\Omega}] - \mathcal{J}[\mathcal{J}\tilde{\Omega}, \Omega]$$
$$[\mathcal{J}\Omega, \mathcal{J}\tilde{\Omega}] - [\Omega, \tilde{\Omega}] - \mathcal{J}[\mathcal{J}\Omega, \tilde{\Omega}] - \mathcal{J}[\Omega, \mathcal{J}\tilde{\Omega}] = 0,$$

which shows that the Nijenhuis tensor $[\mathcal{J}, \mathcal{J}] = 0$.

5.3. Remark. The almost complex structure $\mathcal{J}_2$ was introduced in quite different way by N. Hitchin in [H]. He also proved the integrability and some other properties of $\mathcal{J}_2$.

6. Orbit of forms of the tangent type

Here we shall investigatethe the last orbit U_0, which represents a submanifold of codimension 1 in $\Lambda^3 V^*$. Let $\zeta \in U_0$ be arbitrary point, and let us denote $V_0(\zeta) = \Delta^2(\zeta)$. We shall introduce three subspaces $\mathcal{D}_i(\zeta) \subset V$, $i = 1, 2, 3$ in the following way:

$$\mathcal{D}_i(\zeta) = \{\Omega \in T_\zeta U_0; \Omega(v_1, v_2, v_3) = 0 \text{ if the vectors } v_1, \dots, v_i \text{ belong to } V_0(\zeta)\}.$$

It is easy to verify that $\dim \mathcal{D}_1 = 1$, $\dim \mathcal{D}_2 = 10$, $\dim \mathcal{D}_3 = 19$. Moreover, it is obvious that

$$\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3.$$

We describe first the tangent spaces to the orbit U_0. It is obvious that the projection

$$\pi_\zeta : GL(6, \mathbb{R}) \to U_0, \quad \pi_\zeta(\varphi) = \varphi^* \zeta$$

admits a smooth local section σ defined on an open neighborhood W of ζ and such that $\sigma(\zeta) = 1$. For any $\omega \in W$ we have then

$$\omega = \sigma(\omega)^* \zeta.$$

Let $\gamma : (-\varepsilon, \varepsilon) \to W$ be a smooth curve such that $\gamma(0) = \zeta$. We have then

$$\gamma(t) = \sigma(\gamma(t))^* \zeta$$
$$\gamma(t)(v_1, v_2, v_3) = \zeta(\sigma(\gamma(t))v_1, \sigma(\gamma(t))v_2, \sigma(\gamma(t))v_3),$$

where $v_1, v_2, v_3 \in V$ are arbitrary. Differentiating the last equality at $t = 0$, we get

$$\Omega(v_1, v_2, v_3) = \zeta(Av_1, v_2, v_3) + \zeta(v_1, Av_2, v_3) + \zeta(v_1, v_2, Av_3),$$

where $\Omega = (d/dt)_{t=0}\gamma(t)$ and $A = (d/dt)_{t=0}\sigma(\gamma(t))$.

6.1. Proposition. *There is $T_\zeta U_0 = \mathcal{D}_3(\zeta)$.*

Proof. If $\Omega \in T_\zeta U_0$, then according to the above formula there is $\Omega \in \mathcal{D}_3(\zeta)$ because $\zeta(v, v', v'') = 0$ if two entries belong to $V_0(\zeta)$. We have therefore $T_\zeta U_0 \subset \mathcal{D}_3(\zeta)$. Because $\dim T_\zeta U_0 = 19$ and $\dim \mathcal{D}_3(\zeta) = 19$, we get $T_\zeta U_0 = \mathcal{D}_3(\zeta)$.

It is obvious that it makes no sense to use in the future the notation $\mathcal{D}_3(\zeta)$. The following lemma can be easily verified for the form ω_0. But then it necessarily holds for any form $\zeta \in U_0$

6.2. Lemma. *There is*

$$\mathcal{D}_2(\zeta) = \{\Omega \in T_\zeta U_0; \Omega \wedge (\iota_v \zeta) = 0 \textit{ for every } v \in V_0(\zeta)\} =$$
$$= \{\Omega \in T_\zeta U_0; \Omega \wedge \beta \wedge \beta' = 0 \textit{ for any } \beta, \beta \in V^* \textit{such that } \beta | V_0(\zeta) = \beta' | V_0(\zeta) = 0\}.$$

On U_0 we have the trivial 6-dimensional vector bundle $\mathcal{V}$ with fiber V, and we can define a 3-dimensional vector subbundle $\mathcal{V}_0$ whose fiber at ζ is $V_0(\zeta)$. We denote $\mathcal{W}$ the 3-dimensional quotient vector bundle $\mathcal{V}/\mathcal{V}_0$. Moreover, assigning to each point $\zeta \in U_0$ the vector space $\mathcal{D}_i(\zeta)$, we obtain over U_0 a vector bundle $\mathcal{D}_i$, $i = 1, 2$.

In other words we have two distributions $\mathcal{D}_1 \subset \mathcal{D}_2 \subset TU_0$. Furthermore, we have on U_0 an everywhere non-zero vector field ω defined by the formula $\omega_\zeta = \zeta$, i. e. assigning to a point $\zeta \in U_0$ the vector ζ. This vector field ω lies in the distribution $\mathcal{D}_2$. It is easy to see that the 1-dimensional distribution $\mathcal{I}$ generated by the vector field ω and the 1-dimensional distribution $\mathcal{D}_1$ are transversal.

Fixing a volume form $\theta_0 \in \Lambda^6 V^*$, we get for each $\zeta \in U_0$ a tangent structure $F(\zeta)$. Namely, this tangent structure can be determined by the formula

$$(\iota_v \zeta) \wedge \zeta = \iota_{F(\zeta)v} \theta_0.$$

For any 3-form $\Omega \in \Lambda^3 V^*$ we can then define

$$(D_{F(\zeta)}\Omega)(v_1, v_2, v_3) = \Omega(F(\zeta)v_1, v_2, v_3) + \Omega(v_1, F(\zeta)v_2, v_3) + \Omega(v_1, v_2, F(\zeta)v_3).$$

It is obvious that if $\Omega \in T_\zeta U_0$, then also $D_F\Omega \in T_\zeta U_0$. Consequently, on $T_\zeta U_0$ we can define an endomorphism $\mathcal{N}(\zeta)$ by the formula $\mathcal{N}(\zeta) = D_{F(\zeta)}$. In this way we get on U_0 a tensor field $\mathcal{N}$ of type $(1,1)$. It is easy to see that $\mathcal{N}^3 = 0$.

Our main aim in this section will be to prove the following proposition.

6.3. Proposition. *On U_0 we have the following chain of distributions:*

$$\operatorname{im}\mathcal{N}^2 \subset \ker\mathcal{N} \subset \operatorname{im}\mathcal{N} \subset \ker\mathcal{N}^2,$$

where $\operatorname{im}\mathcal{N}^2 = \mathcal{D}_1$ *and* $\operatorname{im}\mathcal{N} = \mathcal{D}_2$. *The distributions* $\operatorname{im} CalN^2$, $\ker\mathcal{N}$, *and* $\operatorname{im}\mathcal{N}$ *are integrable. The distribution* $\ker\mathcal{N}^2$ *is not integrable.*

6.4. Remark. If $A \in End(V)$ is arbitrary we can define $D_A\Omega$ for any $\Omega \in \Lambda^k V^*$ by the formula

$$(D_F\Omega)(v_1, \dots, v_k) = \sum_{i=1}^{k} \Omega(v_1, \dots, v_{i-1}, Av_i, v_{i+1}, \dots, v_k).$$

It is well known that D_A is a derivation on the graded algebra $\Lambda^* V^*$.

We shall first investigate the subspace $\operatorname{im}\mathcal{N}^2$. Let $\Omega \in \operatorname{im}\mathcal{N}^2(\zeta)$. If $\Omega = \mathcal{N}^2(\zeta)\tilde{\Omega}$, then we have

$$\Omega(v_1, v_2, v_3) = 2(\tilde{\Omega}(Fv_1, Fv_2, v_3) + \tilde{\Omega}(Fv_1, v_2, Fv_3) + \tilde{\Omega}(v_1, Fv_2, Fv_3)),$$

where $F = F(\zeta)$. It is easy to see that if one of the entries v_1, v_2, v_3 belongs to $V_0(\zeta)$, then $\Omega(v_1, v_2, v_3) = 0$, or in other words, $\Omega \in \mathcal{D}_1(\zeta)$. Because obviously $\operatorname{im}\mathcal{N}^2 \neq 0$, we get easily the following lemma. (Notice that $\dim \operatorname{im}\mathcal{N}^2 = 1$.)

6.5. Proposition. *There is* $\operatorname{im}\mathcal{N}^2 = \mathcal{D}_1$ *and* $\operatorname{im}\mathcal{N}^2 \subset \ker\mathcal{N}$. *The distribution* $\operatorname{im}\mathcal{N}^2 \subset TU_0$ *is integrable.*

Next, we shall consider the subspace $\mathcal{D}_2(\zeta)$. It is obvious that for any $\Omega \in \mathcal{D}_2(\zeta)$ the correspondence $v \in V_0(\zeta) \mapsto \iota_v\Omega$ defines a homomorphism

$$\kappa_\Omega : V_0(\zeta) \to \Lambda^2 W(\zeta)^*.$$

We have obvious formulas

$$\kappa_{\Omega+\tilde{\Omega}} = \kappa_\Omega + \kappa_{\tilde{\Omega}}, \quad \kappa_{a\Omega} = a\kappa_\Omega$$

for any $\Omega, \tilde{\Omega} \in \mathcal{D}_2(\zeta)$ and any $a \in \mathbb{R}$. Using the isomorphism $\kappa_\zeta : V_0(\zeta) \to \Lambda^2 W(\zeta)^*$, we can define a homomorphism

$$k_\Omega : \mathcal{D}_2(\zeta) \to End(V_0(\zeta)), \quad k_\Omega(v) = \kappa_\zeta^{-1}\kappa_\Omega(v).$$

It is easy to see that $\ker k_\Omega = \mathcal{D}_1(\zeta)$. Consequently, we get a monomorphism

$$K_\Omega : \mathcal{D}_2(\zeta)/\mathcal{D}_1(\zeta) \to End(V_0(\zeta)).$$

Because $\dim \mathcal{D}_2(\zeta)/\mathcal{D}_1(\zeta) = \dim End(V_0(\zeta)) = 9$, we can see that K_Ω is an isomorphism.

6.6. Proposition. *There is* $\operatorname{im} \mathcal{N} = \mathcal{D}_2$ *and* $\dim \operatorname{im} \mathcal{N} = 10$.

Proof. If $\Omega = \mathcal{N}(\zeta)\hat{\Omega}$, where $\hat{\Omega} \in T_\zeta U_0$, we have

$$\Omega(v_1, v_2, v_3) = \hat{\Omega}(F(\zeta)v_1, v_2, v_3) + \hat{\Omega}(v_1, F(\zeta)v_2, v_3) + \hat{\Omega}(v_1, v_2, F(\zeta)v_3),$$

and it is obvious that $\Omega \in \mathcal{D}_2(\zeta)$. This shows that $\operatorname{im} \mathcal{N} \subset \mathcal{D}_2$.

Conversely, let us assume that $\Omega \in \mathcal{D}_2(\zeta)$. We choose a basis v_1, v_2, v_3 of $V_0(\zeta)$, and we denote $\pi(\zeta) : V \to W(\zeta)$ the projection. Because $\Omega \in \mathcal{D}_2(\zeta)$, there exist 2-forms $\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3 \in \Lambda^2 W(\zeta)^*$ such that

$$\iota_{v_i}\Omega = \pi(\zeta)^*\tilde{\Omega}_i, \quad i = 1, 2, 3.$$

Let us take now 1-forms $\beta_1, \beta_2, \beta_3 \in V^*$ such that $\beta_i(v_j) = \delta_{ij}$. We shall consider a 3-form

$$\hat{\Omega} = \sum_{i=1}^{3} \beta_i \wedge \pi(\zeta)^*\tilde{\Omega}_i.$$

Now we can easily see that $\iota_v(\Omega - \hat{\Omega}) = 0$ for any $v \in V_0(\zeta)$, or in other words $\Omega - \hat{\Omega} \in \mathcal{D}_1 = \operatorname{im} \mathcal{N}^2$. This means that there is a 3-form $\bar{\Omega} \in T_\zeta U_0$ such that $\Omega - \hat{\Omega} = \mathcal{N}^2(\zeta)\bar{\Omega}$.

Let us consider the monomorphism $\pi(\zeta)^* : \Lambda^* W(\zeta)^* \to \Lambda^* V^*$. It is easy to see that $\pi(\zeta)^* W(\zeta)^*$ has a basis $D_{F(\zeta)}\beta_1, D_{F(\zeta)}\beta_2, D_{F(\zeta)}\beta_3$, and that

$$D^2_{F(\zeta)}\beta_1 = D^2_{F(\zeta)}\beta_2 = D^2_{F(\zeta)}\beta_3 = 0.$$

It is obvious that any 2-form $\Omega' \in \pi(\zeta)^*\Lambda^2 W(\zeta)^*$ belongs to $\operatorname{im} D^2_{F(\zeta)}$. Consequently, we can find 2-forms $\Omega'_1, \Omega'_2, \Omega'_3$ such that

$$\pi(\zeta)^*\tilde{\Omega}_i = D^2_{F(\zeta)}\Omega'_i.$$

We have then

$$
\begin{aligned}
\hat{\Omega} &= \sum_{i=1}^{3} \beta_i \wedge \pi(\zeta)^* \tilde{\Omega}_i = \sum_{i=1}^{3} \beta_i \wedge D^2_{F(\zeta)} \Omega'_i = \\
&= \sum_{i=1}^{3} D_{F(\zeta)}(\beta_i \wedge D_{F(\zeta)}\Omega'_i) - \sum_{i=1}^{3} D_{F(\zeta)}\beta_i \wedge D_{F(\zeta)}\Omega'_i = \\
&= D_{F(\zeta)} \sum_{i=1}^{3} \beta_i \wedge D_{F(\zeta)}\Omega'_i - \sum_{i=1}^{3} D_{F(\zeta)}(D_{F(\zeta)}\beta_i \wedge \Omega'_i) = \\
&= D_{F(\zeta)} \sum_{i=1}^{3} (\beta_i \wedge D_{F(\zeta)}\Omega'_i - D_{F(\zeta)}\beta_i \wedge \Omega'_i).
\end{aligned}
$$

Now we can see that $\Omega \in \operatorname{im} \mathcal{N}(\zeta)$, which finishes the proof.

6.7. Proposition. *There is the inclusion* $\ker \mathcal{N} \subset \operatorname{im} \mathcal{N}$.

Proof. Let $\Omega \in \ker \mathcal{N}(\zeta)$. Then we have (we write F instead of $F(\zeta)$)

$$\Omega(Fv_1, v_2, v_3) + \Omega(v_1, Fv_2, v_3) + \Omega(v_1, v_2, Fv_3) = 0.$$

Using this relation we get

$$
\begin{aligned}
\Omega(Fv_1, Fv_2, v_3) &= -\Omega(v_1, F^2v_2, v_3) - \Omega(v_1, Fv_2, Fv_3) = -\Omega(v_1, Fv_2, Fv_3) \\
\Omega(Fv_1, Fv_2, v_3) &= -\Omega(F^2v_1, v_2, v_3) - \Omega(Fv_1, v_2, Fv_3) = -\Omega(Fv_1, v_2, Fv_3)
\end{aligned}
$$

Adding these two relations, we obtain

$$
\begin{gathered}
2\Omega(Fv_1, Fv_2, v_3) = -\Omega(v_1, Fv_2, Fv_3) - \Omega(Fv_1, v_2, Fv_3), \\
\Omega(Fv_1, Fv_2, v_3) = -\Omega(Fv_1, Fv_2, v_3) - \Omega(Fv_1, v_2, Fv_3) - \Omega(v_1, Fv_2, Fv_3) = \\
= -\frac{1}{2} D^2_F \Omega(v_1, v_2, v_3) = 0,
\end{gathered}
$$

which shows that $\Omega \in \mathcal{D}_2(\zeta)$.

6.8. Proposition. *Let* $\zeta \in U_0$. *Then* $\Omega \in T_\zeta U_0$ *belongs to* $\ker \mathcal{N}^2$ *if and only if* $\zeta \wedge \Omega = 0$. *Moreover* $\dim \ker \mathcal{N}^2 = 18$.

Proof. Let us choose vectors $v, v', v'' \in V$ such that $Fv, Fv', Fv'', v, v', v''$ is a basis of V. (We denote for simplicity $F = F(\zeta)$.) We shall consider the value $(\zeta \wedge \Omega)(Fv, Fv', Fv'', v, v', v'')$. (We recall that $\zeta(w, w', \cdot) = 0$ if $w, w' \in V_0(\zeta)$, $\zeta(w, Fw, \cdot) = 0$ for any $w \in V$, and $\Omega | V_0(\zeta) = 0$.) We get

$$
\begin{gathered}
(\zeta \wedge \Omega)(Fv, Fv', Fv'', v, v', v'') = \zeta(Fv, v', v'')\Omega(Fv', Fv'', v) + \\
+ \zeta(Fv', v, v'')\Omega(Fv, Fv'', v') + \zeta(Fv'', v, v')\Omega(Fv, Fv', v'') = \\
= \zeta(Fv, v', v'')[\Omega(Fv, Fv', v'') + \Omega(Fv, v', Fv'') + \Omega(v, Fv', Fv'')].
\end{gathered}
$$

Because $\zeta(Fv, v', v'') \neq 0$ the first assertion easily follows. Now it is obvious that $\dim \ker \mathcal{N}^2 = 18$.

6.9. Proposition. *There is* $\operatorname{im}\mathcal{N} \subset \ker \mathcal{N}^2$.

Proof. If $\Omega \in \operatorname{im}\mathcal{N}(\zeta) = \mathcal{D}_2(\zeta)$ then obviously $\zeta \wedge \Omega = 0$.

On the trivial vector bundle $\mathcal{V}$ with fiber V over U_0 we introduce a linear connection ∇. For any vector field Ω on U_0 and any section S of $\mathcal{V}$ we define $\nabla_\Omega S = \Omega S$. Obviously, ∇ induces a linear connection on every exterior power $\Lambda^k V^*$, which will be denoted by the same symbol. It is obvious that the same formula $\bar{\nabla}_{\bar{\Omega}}\bar{S} = \bar{\Omega}\bar{S}$, where $\bar{S}$ is a section of the trivial vector bundle $\bar{\mathcal{V}}$ with fiber V over $\Lambda^3 V^*$, extends the connection ∇ to the whole vector space $\Lambda^3 V^*$. The connection $\bar{\nabla}$ induces again a linear connection on the vector bundle $\Lambda^k \bar{\mathcal{V}}^*$, which will be denoted again by the symbol $\bar{\nabla}$. Let Ω_1 and Ω_2 be (local) vector fields on U_0, and let $\bar{\Omega}_1$ and $\bar{\Omega}_2$ be their (local) extensions. Because the connection $\bar{\nabla}$ is flat, we have $\bar{\nabla}_{\bar{\Omega}_1}\bar{\Omega}_2 - \bar{\nabla}_{\bar{\Omega}_2}\bar{\Omega}_1 = [\bar{\Omega}_1, \bar{\Omega}_2]$. Restricting this formula to the submanifold U_0, we obtain the formula

$$\nabla_{\Omega_1}\Omega_2 - \nabla_{\Omega_2}\Omega_1 = [\Omega_1, \Omega_2],$$

which will be needed in the sequel.

6.10. Lemma. *Let S be a section of the subbundle $\mathcal{V}_0$, and let Ω be a vector field on U_0 lying in* $\operatorname{im}\mathcal{N}$. *Then $\nabla_\Omega S$ is also a section of the subbundle $\mathcal{V}_0$.*

Proof. Because S is a section of the subbundle $\mathcal{V}_0$, we have the relation $(\iota_S\omega)\wedge\omega = 0$. Applying to this relation ∇_Ω, we obtain

$$(\iota_{\nabla_\Omega S}\omega)\wedge\omega + (\iota_S\Omega)\wedge\omega + (\iota_S\omega)\wedge\Omega = 0.$$

It is easy to see that the second term vanishes. The last term vanishes by virtue of Lemma 6.3. Consequently, we obtain $(\iota_{\nabla_\Omega S}\omega)\wedge\omega = 0$, which shows that $\nabla_\Omega S$ is a section of $\mathcal{V}_0$.

6.11. Remark. The previous lemma shows that the connection ∇ on $\mathcal{V}$ induces a partial connection on $\mathcal{V}_0$, which we shall denote by the same symbol. This partial connection determines the covariant derivative ∇_Ω only for vector the fields Ω lying in $\operatorname{im}\mathcal{N}$. This partial connection induces a partial connection on the vector bundle $\mathcal{W}$ and on any exterior power of the vector bundles $\mathcal{V}_0$ and $\mathcal{W}$. Moreover, if $\tilde{\Omega}$ is a vector field on U_0 (i. e. a section of $\Lambda^3\mathcal{V}^*$ such that $\tilde{\Omega}|\mathcal{V}_0 = 0$), then for any vector field Ω lying in $\operatorname{im}\mathcal{N}$ and any three sections S_1, S_2, S_3 of $\mathcal{V}_0$ we have

$$\begin{gathered}
\tilde{\Omega}(S_1, S_2, S_3) = 0 \\
\nabla_\Omega(\tilde{\Omega}(S_1, S_2, S_3)) = 0 \\
(\nabla_\omega\tilde{\Omega})(S_1, S_2, S_3) + \tilde{\Omega}(\nabla_\Omega S_1, S_2, S_3) + \tilde{\Omega}(S_1, \nabla_\Omega S_2, S_3) + \tilde{\Omega}(S_1, S_2, \nabla_\Omega S_3) = 0 \\
(\nabla_\Omega\tilde{\Omega})(S_1, S_2, S_3) = 0,
\end{gathered}$$

which shows that the partial connection ∇ induces a partial connection (again denoted by the same symbol) on TU_0. Because the original connection on $\mathcal{V}$ is flat, we have for any two vector fields Ω and $\tilde{\Omega}$ lying in $\operatorname{im}\mathcal{N}$

$$\nabla_\Omega\tilde{\Omega} - \nabla_{\tilde{\Omega}}\Omega = [\Omega, \tilde{\Omega}].$$

6.12. Proposition. *The distribution* $\operatorname{im}\mathcal{N}$ *is integrable.*

Proof. According to Proposition 6.6 there is $\operatorname{im}\mathcal{N} = \mathcal{D}_2$. Let us take two vector fields $\Omega, \tilde{\Omega}$ lying in $\mathcal{D}_2$, and three sections S_1, S_2, S_3 of $\mathcal{V}$ such that S_1 and S_2 lie in $\mathcal{V}_0$. Then we have

$$(\nabla_\Omega \tilde{\Omega})(S_1, S_2, S_3) = \nabla_\Omega(\tilde{\Omega}(S_1, S_2, S_3)) - \\ -\tilde{\Omega}(\nabla_\Omega S_1, S_2, S_3) - \tilde{\Omega}(S_1, \nabla_\Omega S_2, S_3) - \tilde{\Omega}(S_1, S_2, \nabla_\Omega S_3) = 0$$

according to Lemma 6.10. This shows that $\nabla_\Omega \tilde{\Omega}$ lies in $\mathcal{D}_2$. Now, it is obvious that $[\Omega, \tilde{\Omega}] = \nabla_\Omega \tilde{\Omega} - \nabla_{\tilde{\Omega}} \Omega$ lies in $\mathcal{D}_2$.

6.13. Proposition. $\ker \mathcal{N} = \{\Omega \in \operatorname{im}\mathcal{N}; \operatorname{Tr} k(\Omega) = 0\}$ *and* $\dim \ker \mathcal{N} = 9$.

Proof. We shall denote for simplicity $V_0 = V_0(\zeta)$, $F = F(\zeta)$, $W = W(\zeta)$, $\pi = \pi(\zeta)$. Let us notice first that for each endomorphism $A \in End(V_0)$ there exists an endomorphism $B \in End(V)$ (not uniquely determined) such that

$$AF = FB \quad \text{and} \quad BV_0 \subset V_0.$$

Moreover, any endomorphism B with these properties induces an endomorphism $\tilde{B} \in End(W)$ and $\operatorname{Tr} \tilde{B} = \operatorname{Tr} A$.

Let us take now a 3-form $\Omega \in \operatorname{im}\mathcal{N}(\zeta) = \mathcal{D}_2(\zeta)$. We have

$$(\mathcal{N}(\zeta)\Omega)(v, v', v'') = \Omega(Fv, v', v'') + \Omega(v, Fv', v'') + \Omega(v, v', Fv'').$$

It is easy to see that $\mathcal{N}(\zeta)\Omega \in \mathcal{D}_1(\zeta)$, and consequently there exists a uniquely determined 3-form $\tilde{\Omega} \in \Lambda^3 W^*$ such that $\mathcal{N}(\zeta)\Omega = \pi^*\tilde{\Omega}$. Similarly, there is a 3-form $\tilde{\zeta} \in \Lambda^3 W^*$ such that $\mathcal{N}(\zeta)\zeta = \pi^*\tilde{\zeta}$. We recall that the homomorphism $\pi^* : \Lambda^3 W^* \to \Lambda^3 V^*$ is a monomorphism. Consequently $\tilde{\zeta} \neq 0$.

Let us take now $A = k(\zeta)$. Obviously for any $v, v', v'' \in V$ we have

$$\zeta(AFv, v', v'') = \Omega(Fv, v', v''), \\ \zeta(v, AFv', v'') = \Omega(v, Fv', v''), \\ \zeta(v, v', AFv'') = \Omega(v, v', Fv'').$$

Then we get

$$(\mathcal{N}(\zeta)\Omega)(v, v', v'') = \Omega(Fv, v', v'') + \Omega(v, Fv', v'') + \Omega(v, v', Fv'') = \\ = \zeta(AFv, v', v'') + \zeta(v, AFv', v'') + \zeta(v, v', AFv'') = \\ = \zeta(FBv, v', v'') + \zeta(v, FBv', v'')\zeta(v, v', FBv'') = \\ = (1/3)[\zeta(FBv, v', v'') + \zeta(Bv, Fv', v'') + \zeta(Bv, v', Fv'')] + \\ +(1/3)[\zeta(Fv, Bv', v'') + \zeta(v, FBv', v'') + \zeta(v, Bv', Fv'')] + \\ +(1/3)[\zeta(Fv, v', Bv'') + \zeta(v, Fv', Bv'') + \zeta(v, v', FBv'')] = \\ = (1/3)\tilde{\zeta}(\tilde{B}[v], [v'], [v'']) + (1/3)\tilde{\zeta}([v], \tilde{B}[v'], [v'']) + (1/3)\tilde{\zeta}([v], [v'], \tilde{B}[v'']) = \\ = (1/3)\operatorname{Tr}(\tilde{B})\tilde{\zeta}([v], [v'], [v'']) = (1/3)\operatorname{Tr}(A)\zeta(v, v', v''),$$

which shows that $\mathcal{N}(\zeta)\Omega = 0$ if and only if $\operatorname{Tr}(A) = 0$.

6.14. Lemma. *Let M be a differentiable manifold, and let ξ be an n-dimensional differentiable vector bundle over M endowed with a linear connection ∇. Let A be an endomorphism of the vector bundle ξ, i. e. a section of the vector bundle $\xi^* \otimes \xi$. Then for any vector field X on M we have*

$$\operatorname{Tr}(\nabla_X A) = X \operatorname{Tr}(A).$$

Proof. Let us choose (at least locally) a non-zero n-form ε on ξ. Then for any vector fields $X_1, \ldots, X_n$ we have

$$\sum_{i=1}^{n} \varepsilon(X_1, \ldots, X_{i-1}, AX_i, X_{i+1}, \ldots, X_n) = \operatorname{Tr}(A) \cdot \varepsilon(X_1, \ldots, X_n).$$

Let X be a vector field on M. Applying ∇_X to the above equality, we obtain

$$\sum_{i=1}^{n} \varepsilon(X_1, \ldots, X_{i-1}, (\nabla_X A)X_i, X_{i+1}, \ldots, X_n) = (X \operatorname{Tr}(A)) \cdot \varepsilon(X_1, \ldots, X_n),$$

which implies the desired equality.

6.15. Proposition. *The distribution* $\ker \mathcal{N}$ *is integrable.*

Proof. Let Ω and $\tilde{\Omega}$ be two vector fields lying in the distribution $\ker \mathcal{N}$. We denote $A = k_\Omega$ and $\tilde{A} = k_{\tilde{\Omega}}$. According to the previous result there is $\operatorname{Tr}(A) = \operatorname{Tr}(\tilde{A}) = 0$. For any section S of $\mathcal{V}_0$ and any constant sections S', S'' of $\mathcal{V}$ we have

$$\omega(AS, S', S'') = \Omega(S, S', S''), \quad \omega(\tilde{A}S, S', S'') = \tilde{\Omega}(S, S', S'').$$

Applying ∇_Ω to the second equality we obtain

$$\begin{gathered}
(\nabla_\Omega \omega)(\tilde{A}S, S', S'') + \omega((\nabla_\Omega \tilde{A})S, S', S'') + \omega(\tilde{A}\nabla_\Omega S, S', S'') = \\
= (\nabla_\Omega \tilde{\Omega})(S, S', S'') + \tilde{\Omega}(\nabla_\Omega S, S', S'') \\
\Omega(\tilde{A}S, S', S'') + \omega((\nabla_\Omega \tilde{A})S, S', S'') = (\nabla_\Omega \tilde{\Omega})(S, S', S'') \\
\omega(A\tilde{A}S, S'.S'') + \omega((\nabla_\Omega \tilde{A})S, S', S'') = (\nabla_\Omega \tilde{\Omega})(S, S', S'') \\
(\nabla_\Omega \tilde{\Omega})(S, S', S'') = \omega((A\tilde{A} + \nabla_\Omega \tilde{A})S, S', S'').
\end{gathered}$$

Similarly we obtain

$$(\nabla_{\tilde{\Omega}} \Omega)(S, S', S'') = \omega((\tilde{A}A + \nabla_{\tilde{\Omega}} A)S, S', S'').$$

Substracting the last two equalities we have

$$[\Omega, \tilde{\Omega}](S, S', S'') = (\nabla_\Omega \tilde{\Omega} - \nabla_{\tilde{\Omega}} \Omega)(S, S', S'') = \omega(([A, \tilde{A}] + \nabla_\Omega \tilde{A} - \nabla_{\tilde{\Omega}} A)S, S', S''),$$

which shows that

$$k_{[\Omega, \tilde{\Omega}]} = [A, \tilde{A}] + \nabla_\Omega \tilde{A} - \nabla_{\tilde{\Omega}} A.$$

On any integral submanifold of the distribution $\operatorname{im} \mathcal{N}$ we have

$$\operatorname{Tr}([A, \tilde{A}] + \nabla_\Omega \tilde{A} - \nabla_{\tilde{\Omega}} A) = 0 + \Omega \operatorname{Tr}(\tilde{A}) - \tilde{\Omega} \operatorname{Tr}(A) = 0.$$

This finishes the proof.

6.16. Proposition. *The distribution* $\ker \mathcal{N}^2$ *is not integrable.*

Proof. Let Ω and $\tilde{\Omega}$ be two vector fields on U_0 lying in $\ker \mathcal{N}^2$. We shall apply the vector field Ω to the relation $\tilde{\Omega} \wedge \omega = 0$. We get

$$(\nabla_\Omega \tilde{\Omega}) \wedge \omega + \tilde{\Omega} \wedge \Omega = 0.$$

Interchanging Ω and $\tilde{\Omega}$ and substracting the two relations, we obtain

$$[\Omega, \tilde{\Omega}] \wedge \omega + \tilde{\Omega} \wedge \Omega - \Omega \wedge \tilde{\Omega} = 0$$
$$[\Omega, \tilde{\Omega}] \wedge \omega = 2\Omega \wedge \tilde{\Omega}.$$

Let us choose now vectors $\alpha, \tilde{\alpha} \in T_{\omega_0} U_0$ as follows:

$$\alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_5, \quad \tilde{\alpha} = \alpha_3 \wedge \alpha_4 \wedge \alpha_6.$$

It is easy to verify that $\Omega, \tilde{\Omega} \in \ker \mathcal{N}^2(\omega_0)$. We choose now vector fields Ω and $\tilde{\Omega}$ in such a way that they lie in $\ker \mathcal{N}^2$ and $\Omega_{\omega_0} = \alpha$ and $\tilde{\Omega}_{\omega_0} = \tilde{\alpha}$. According to the above formula we have then

$$[\Omega, \tilde{\Omega}]_{\omega_0} \wedge \omega_0 = 2\alpha \wedge \tilde{\alpha} \neq 0,$$

which shows that the vector field $[\Omega, \tilde{\Omega}]$ does not lie in $\ker \mathcal{N}^2$.

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